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Feynman propagator for time-dependent Lagrangians possessing an invariant quadratic in momentum

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Abstract. A class of time-dependent classical Lagrangians possessing an invariant quadratic in momentum is considered from a quantal point of view. Quantum mechanics is introduced through the Feynman propagator defined as a path integral involving the classical action. It is shown, without carrying out an explicit path integration, that the propagator for such a time-dependent system is related to the propagator of an associated time-independent problem. The expansion of the propagator in terms of the eigenfunctions of the invariant operator is derived and the equivalence of the present theory to that of Lewis and Reisenfeld is discussed. Explicit analytic forms of propagators are obtained for some cases to illustrate the application of the present approach.

1. Introduction

The problem of obtaining exact invariants of motion for certain time-dependent systems has received a great deal of attention. Since Lewis (1967, 1968) rederived a previously known (Courant and Snyder 1958) quadratic invariant for the variable frequency oscillator, considerable progress has been made in deriving similar invariants for more general time-dependent systems. This has been achieved by an application of Noether's theorem to the underlying Lagrangian (Lutzky 1978, 1980, Ray and Reid 1979a, b, c, 1982, Reid and Ray 1980, Ray 1981), through group theoretic methods applied to the Hamiltonian (Gunther and Leach 1977, Prince and Eliezer 1980), by means of canonical transformations (Sarlet 1978, Lewis and Leach 1982) or by a direct approach (Lewis and Leach 1982). Apart from their intrinsic mathematical interest, the invariants have evoked attention because of their use in discussing several physical problems (Courant and Snyder 1958, Colegrave and Abdalla 1981, Lewis and Leach 1983).

Another important application of invariants is towards solving the time-dependent quantum mechanical problems. The first step in this direction was taken by Lewis and Riesenfeld (1969) who showed that for a quantal system characterised by a time-dependent Hamiltonian $\hat{H}(t)$ and a Hermitian invariant $\hat{I}(t)$ the general solution of the time-dependent Schrödinger equation

$$i\hbar\, \partial\Psi(q, t)/\partial t = \hat{H}(t)\Psi(q, t) \tag{1.1}$$

is given by

$$\Psi(q, t) = \sum_n C_n\, e^{i\alpha_n(t)}\psi_n(q, t). \tag{1.2}$$

Here $\psi_n(q, t)$ are the normalised eigenfunctions of the invariant operator:

$$\hat{I}\psi_n(q, t) = \lambda_n \psi_n(q, t), \tag{1.3}$$

where the eigenvalues are time independent. The expansion coefficients C_n are constants while the time-dependent phases $\alpha_n(t)$ are to be determined from the equation

$$\hbar\, \mathrm{d}\alpha_n(t)/\mathrm{d}t = \langle \psi_n | \mathrm{i}\hbar\, \partial/\partial t - \hat{H} | \psi_n \rangle. \tag{1.4}$$

Lewis and Riesenfeld (1969) employed this result to obtain quantal solutions for a time-dependent oscillator and a charged particle in a time-varying electromagnetic field. Recently, Hartley and Ray (1981, 1982) have applied this technique to derive a quantum mechanical superposition law for more generalised time-dependent systems.

In this paper, we consider the alternative route from classical to quantum mechanics via the Feynman propagator. The propagator requires only the knowledge of the classical Lagrangian and has the added advantage that quantum superposition is already built into it. The questions that we pose and attempt partially to answer are the following. Given a classical Lagrangian that admits an invariant, what form does the propagator take? In particular, what is the role played by the invariant in this approach to quantisation? Explicit path integral calculations for time-dependent problems have shown that propagators admit expansions in terms of the eigenfunctions of the invariant operator (Khandekar and Lawande 1975, 1979). Apart from this, it is difficult to provide general answers to the above questions. Clearly, the existence of an invariant imposes certain conditions on the admissible forms of the potential $V(q, t)$ in the Lagrangian and this fact may simplify the derivation of the propagator.

A great deal of simplification arises if we assume that the form of the invariant is quadratic in momentum p. We show in § 2, without explicit path integration that in this case, the Feynman propagator for the time-dependent problem is related to the propagator for an associated time-independent problem. This derivation is more general than the short version recently reported (Lawande and Dhara 1983). Furthermore, the expansion of the propagator in terms of the eigenfunctions of the invariant operator and the equivalence of the propagator theory to that of Lewis and Riesenfeld is discussed. Examples where our theory leads to exact analytic propagators are presented in § 3. Finally, some concluding remarks are added in § 4.

2. Feynman propagator

2.1. Derivation of the propagator

The Feynman propagator $K(q'', t''; q', t')$ is the quantum mechanical amplitude for finding a particle at the position q'' at time t'' if the particle had been at q' at an earlier time t'. It is defined as the path integral (Feynman and Hibbs 1965)

$$K(q'', t''; q', t') = \int \exp\left(\frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} L\, \mathrm{d}t\right) \mathscr{D}q(t), \tag{2.1}$$

where L is the classical Lagrangian

$$L(q, \dot{q}, t) = \tfrac{1}{2}\dot{q}^2 - V(q, t) \tag{2.2}$$

while $\mathscr{D}q(t)$ is the usual Feynman path differential measure implying that integrations are over all possible particle paths starting at $q(t') = q'$ and terminating at $q(t'') = q''$.

We assume that the system described by (2.2) possesses an invariant which is quadratic in the momentum p. This will be the case (Lewis and Leach 1982) if and only if the potential is of the form

$$V(q,t)=\left(\frac{\ddot{\rho}\alpha}{\rho}-\ddot{\alpha}\right)q-\frac{\ddot{\rho}q^2}{2\rho}+\frac{1}{\rho^2}F\left(\frac{q-\alpha}{\rho}\right), \tag{2.3}$$

where $\rho(t)$, $\alpha(t)$ and $F((q-\alpha)/\rho)$ are arbitrary functions of their arguments. Incidentally, the associated invariant has the form

$$I(q,p,t)=\tfrac{1}{2}[\rho(p-\dot{\alpha})-\dot{\rho}(q-\alpha)]^2+F((q-\alpha)/\rho). \tag{2.4}$$

Inserting the potential (2.3) in (2.2) and carrying out some algebra, it is possible to rewrite the Lagrangian as

$$L=\mathrm{d}\chi/\mathrm{d}t+L_0. \tag{2.5}$$

Here the new Lagrangian L_0 has the form

$$L_0=\tfrac{1}{2}\rho^2\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{q-\alpha}{\rho}\right)\right]^2-\frac{1}{\rho^2}F\left(\frac{q-\alpha}{\rho}\right) \tag{2.6}$$

while χ is defined as

$$\chi=\frac{\dot{\rho}q^2}{2\rho}+Wq/\rho-G, \tag{2.7}$$

where

$$W=\dot{\alpha}\rho-\alpha\dot{\rho}=\rho^2(\mathrm{d}/\mathrm{d}t)(\alpha/\rho) \tag{2.8}$$

$$G=\tfrac{1}{2}\int^t\rho^2\left[\frac{\mathrm{d}}{\mathrm{d}t}(\alpha/\rho)\right]^2\mathrm{d}t. \tag{2.9}$$

It is now clear from

$$\int_{t'}^{t''}L\,\mathrm{d}t=(\chi(t'')-\chi(t'))+\int_{t'}^{t''}L_0\,\mathrm{d}t \tag{2.10}$$

that the first term on the RHS of (2.10) will lead to a constant phase factor depending on the initial and final positions q' and q'' in the path integral expression (2.1). Hence the path integration for the original Lagrangian L reduces to a path integration for a related Lagrangian L_0. Thus we may write

$$K(q'',t'';q',t)=\exp[(\mathrm{i}/\hbar)(\chi(t'')-\chi(t'))]K_0, \tag{2.11}$$

where

$$K_0=\int\exp\left(\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}L_0\,\mathrm{d}t\right)\mathscr{D}q(t) \tag{2.12}$$

is the new propagator involving the new Lagrangian L_0. Since $\alpha(t)$ is a given function of time it is possible to introduce the new variable

$$Q=q-\alpha \tag{2.13}$$

in the path integration of (2.12). This will merely shift the end points from q' to $Q'=q'-\alpha'$ and q'' to $Q''=q''-\alpha''$ (where primes and double primes imply that the quantities are evaluated at t' and t'' respectively). Thus

$$K_0 = \int \exp\left(\frac{\mathrm{i}}{\hbar}\int L_0\,\mathrm{d}t\right)\mathscr{D}Q(t) \tag{2.14}$$

with

$$L_0 \equiv \tfrac{1}{2}\rho^2\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{Q}{\rho}\right)\right]^2 - \frac{1}{\rho^2}F\left(\frac{Q}{\rho}\right). \tag{2.15}$$

Next, we introduce a new parameter related to time t by

$$\tau(t) = \int^t \rho^{-2}(s)\,\mathrm{d}s$$

so that the action integral in (2.14) takes the form

$$\int_{t'}^{t''} L_0\,\mathrm{d}t = \int_{\tau'}^{\tau''} \bar{L}_0\,\mathrm{d}\tau, \tag{2.16}$$

where the new Lagrangian $\bar{L}_0$ has the form

$$\bar{L}_0 \equiv \bar{L}_0(\xi, \mathrm{d}\xi/\mathrm{d}\tau) = \tfrac{1}{2}(\mathrm{d}\xi/\mathrm{d}\tau)^2 - F(\xi), \tag{2.17}$$

where $\xi = Q/\rho$. It is important to note here that the parameter τ would in turn induce a transformation in the path differential measure $\mathscr{D}Q(t)$. The required transformation of the path differentiable measure as $t \to \tau$ has been considered by Fujiwara (1969) and subsequently used by Lawande and Dhara (1983). It has the form

$$\mathscr{D}Q(t) = (\rho'\rho'')^{-1/2}\mathscr{D}\xi(\tau). \tag{2.18}$$

It therefore follows from (2.11)–(2.18) that the Feynman propagator takes the form

$$K(q'', t''; q', t') = (\rho'\rho'')^{-1/2}\exp[(\mathrm{i}/\hbar)[\chi(t'') - \chi(t')]\bar{K}_0(\xi'', \tau''; \xi', \tau') \tag{2.19a}$$

where

$$\bar{K}_0(\xi'', \tau''; \xi', \tau') = \int \exp\left[\frac{\mathrm{i}}{\hbar}\int_{\tau'}^{\tau''} \bar{L}_0\,\mathrm{d}\tau\right]\mathscr{D}\xi(\tau). \tag{2.19b}$$

We have thus shown that the propagator for a time-dependent problem is related to the propagator of an associated time-independent problem corresponding to the Lagrangian $\bar{L}_0$ in the new space–time (ξ, τ).

We mention here that the above formulation may be easily extended to obtain the propagator for a more general Lagrangian

$$L = \tfrac{1}{2}a(t)\dot{q}^2 - V(q, t)$$

possessing an invariant quadratic in momentum. Physically, $a(t)$ represents either a variable mass (Colegrave and Abdalla 1981, 1982, 1983, Leach 1983) or a frictional force depending linearly on velocity (e.g. Khandekar and Lawande 1979).

2.2. Expansion of the propagator

Further implications of the result (2.19) derived in § 2.1 will be clear if we consider the role played by the invariant in the propagator theory. The classical Hamiltonian corresponding to the Lagrangian $\bar{L}_0$ is

$$\bar{H}_0 = \tfrac{1}{2}p_\xi^2 + F(\xi), \tag{2.20}$$

where p_ξ is the canonical momentum conjugate to the new variable ξ. On the other hand, it is easy to see that the invariant (2.4) when written in terms of the new variable $\xi = (q-\alpha)/\rho$ and τ is identical to $\bar{H}_0$.

The corresponding quantum Hamiltonian $\hat{\bar{H}}_0$ and the invariant operator $\hat{I}_0$ are obtained by writing $p_\xi = -\mathrm{i}\hbar\,\partial/\partial\xi$ in (2.20)

$$\hat{\bar{H}}_0 = (-\tfrac{1}{2}\hbar^2\,\partial^2/\partial\xi^2 + F(\xi)) \equiv \hat{I}_0. \tag{2.21}$$

The propagator $\bar{K}_0(\xi'', \tau''; \xi', \tau')$ then represents the Green function of the Schrödinger equation

$$\mathrm{i}\hbar\,\partial\psi_n(\xi, \tau)/\partial\tau = \hat{\bar{H}}_0\psi_n(\xi, \tau). \tag{2.22}$$

Hence, if the associated stationary problem

$$\hat{\bar{H}}_0\phi_n(\xi) = \lambda_n\phi_n(\xi) \tag{2.23}$$

has a complete set of normalised eigenfunctions $\phi_n(\xi)$ corresponding to eigenvalues λ_n, the propagator $\bar{K}_0$ has the expansion (Feynman and Hibbs 1965)

$$\bar{K}_0(\xi'', \tau''; \xi', \tau') = \sum_n \exp[-(\mathrm{i}/\hbar)\lambda_n(\tau''-\tau')]\phi_n^*(\xi')\phi_n(\xi''). \tag{2.24}$$

Note that the eigenvalues λ_n may be both discrete and continuous and equation (2.24) implies in general a summation over continuous eigenvalues. Inserting the original variables q, t, we see that the propagator (2.19) has the following expansion in terms of the eigenfunctions of the invariant operator:

$$K(q'', t''; q', t') = (\rho'\rho'')^{-1/2}\left[\frac{\mathrm{i}}{\hbar}(\chi(t'') - \chi(t'))\right]$$
$$\times\sum_n \exp\left(-\frac{\mathrm{i}}{\hbar}\lambda_n\int_{t'}^{t''}\frac{\mathrm{d}t}{\rho^2}\right)\phi_n\left(\frac{q''-\alpha''}{\rho''}\right)\phi_n^*\left(\frac{q'-\alpha'}{\rho'}\right). \tag{2.25}$$

In order to compare the propagator approach with Lewis and Riesenfeld theory, consider the quantal Hamiltonian corresponding to the potential (2.3). If we perform a unitary transformation

$$\psi_n' = U\psi_n, \qquad U = \exp\{-(\mathrm{i}/\hbar)[\dot\rho q^2/2\rho + (\dot\alpha\rho - \alpha\dot\rho)q/\rho]\} \tag{2.26}$$

equation (1.3) transforms to

$$\hat{I}'\psi_n'(q, t) = \lambda_n\psi_n'(q, t), \tag{2.27}$$

where

$$\hat{I}' = U\hat{I}U^\dagger. \tag{2.28}$$

This transformed invariant $\hat{I}'$ when expressed in terms of new variables $\xi = (q-\alpha)/\rho$ corresponds to $\hat{I}_0$ or $\hat{\bar{H}}_0$ defined in (2.21) and the corresponding normalised eigenfunctions $\phi_n(\xi) = \sqrt{\rho}\,\psi_n'(q, t)$. Further, employing the unitary transformation, the solution of equation (1.4) can be easily obtained:

$$\alpha_n(t) = -\frac{1}{\hbar}\int^t \mathrm{d}t\left(\frac{\lambda_n}{\rho^2} + \frac{W^2}{2\rho^2}\right) \tag{2.29}$$

which appears naturally in expansion (2.25). It follows that the result expressed in (2.25) is thus equivalent to the Lewis and Riesenfeld (1969) theory. It must be remarked here, however, that, our present derivation is based on the assumption that the invariant is quadratic in momentum while no such assumption is explicit in the Lewis and Riesenfeld theory.

3. Applications

In this section, we use formula (2.19) to obtain exact propagators for some time-dependent systems possessing invariants quadratic in momentum. The most interesting example is that of a forced harmonic oscillator with time-dependent frequency $\Omega^2(t)$ and force function $f(t)$. Comparing the potential

$$V(q, t) = \tfrac{1}{2}\Omega^2(t)q^2 - f(t)q \tag{3.1}$$

with the general form (2.3) we may set $F = 0$. The time-dependent functions $\rho(t)$ and $\alpha(t)$ are found to obey the equations

$$\ddot{\rho} + \Omega^2(t)\rho = 0 \tag{3.2}$$

$$\ddot{\alpha} + \Omega^2(t)\alpha = f(t). \tag{3.3}$$

Since $F \equiv 0$, the propagator $\bar{K}_0(\xi'', \tau''; \xi', \tau')$ takes the form for a free-particle

$$\bar{K}_0(\xi'', \tau''; \xi', \tau') = [2\pi i\hbar(\tau'' - \tau')]^{-1/2} \exp\left(\frac{i}{2\hbar}\frac{(\xi'' - \xi')^2}{\tau'' - \tau'}\right). \tag{3.4}$$

Inserting (3.4) in the expression (2.19) and recalling the definition (2.7) of $\chi(t)$ and letting $\xi = (q - \alpha)/\rho$ we may write

$$K(q'', t''; q', t') = (c/2\pi i\hbar)^{1/2} \exp[(i/2\hbar)(aq''^2 + bq'^2 - 2cq'q'' + dq'' + eq' + g)], \tag{3.5}$$

where the various coefficients are defined as follows

$$a = (\dot{\rho}'' + c\rho')/\rho'', \tag{3.6}$$

$$b = (-\dot{\rho}' + c\rho'')/\rho', \tag{3.7}$$

$$c = [\rho'\rho''(\tau'' - \tau')]^{-1}, \tag{3.8}$$

$$d = (2/\rho'')[W(t'') - c(\alpha''\rho' - \alpha'\rho'')], \tag{3.9}$$

$$e = (2/\rho')[c(\alpha''\rho' - \alpha'\rho'') - W(t')], \tag{3.10}$$

$$g = c(\alpha''\rho' - \alpha'\rho'')^2/\rho'\rho'' - \int_{t'}^{t''} \frac{W^2(t)}{\rho^2(t)}\,dt. \tag{3.11}$$

In order to evaluate these coefficients we need to use solutions ρ and α of equations (3.2) and (3.3) respectively. We note that a solution of (3.2) is given by

$$\rho(t) = \sigma(t)\cos\beta(t), \tag{3.12}$$

where $\sigma(t)$ satisfy the equation of Pinney (1950)

$$\ddot{\sigma} + \Omega^2(t)\sigma = \omega^2/\sigma^3 \tag{3.13}$$

and

$$\dot{\beta} = \omega/\sigma^2 \tag{3.14}$$

with ω as a constant. A particular solution of (3.3) reads as

$$\alpha(t)=\frac{\sigma(t)}{\omega}\int^{t} H(s)\sin\phi(t,s)\,\mathrm{d}s. \tag{3.15}$$

where the abbreviations

$$\phi(t,s)=\beta(t)-\beta(s)=\omega\int_s^t\frac{\mathrm{d}t'}{\sigma^2(t')}, \tag{3.16}$$

$$H(s)=f(s)\sigma(s) \tag{3.17}$$

are introduced. Inserting the solutions (3.12) and (3.15) in equations (3.6)–(3.11) we obtain after some algebra the expressions:

$$a=\dot\sigma''/\sigma''+(\omega/\sigma''^2)\cot\phi(t'',t'), \tag{3.18}$$

$$b=-\dot\sigma'/\sigma'+(\omega/\sigma'^2)\cot\phi(t'',t'), \tag{3.19}$$

$$c=\omega/\sigma'\sigma''\sin\phi(t'',t'), \tag{3.20}$$

$$d=\frac{2}{\sigma''\sin\phi(t'',t')}\int_{t'}^{t''}H(t)\sin\phi(t,t')\cdot\mathrm{d}t, \tag{3.21}$$

$$e=\frac{2}{\sigma'\sin\phi(t'',t')}\int_{t'}^{t''}H(t)\sin\phi(t'',t)\,\mathrm{d}t, \tag{3.22}$$

$$g=-\frac{2}{\omega\sin\phi(t'',t')}\int_{t'}^{t''}\mathrm{d}t\int_{t'}^{t}\mathrm{d}s\,H(t)H(s)\sin\phi(t'',t)\sin\phi(s,t'). \tag{3.23}$$

Combining these results we arrive at the propagator

$$\begin{aligned}K(q'',t'';q',t')&\\&=\left(\frac{\omega}{2\pi\mathrm{i}\hbar\sigma''\sigma'\sin\phi(t'',t')}\right)^{1/2}\exp\left[\frac{\mathrm{i}}{2\hbar}\left(\frac{\dot\sigma''}{\sigma''}q''^2-\frac{\dot\sigma'}{\sigma'}q'^2\right)\right]\\&\quad\times\exp\left\{\frac{\mathrm{i}\omega}{2\hbar\sin\phi(t'',t')}\left[\left(\frac{q''^2}{\sigma''^2}+\frac{q'^2}{\sigma'^2}\right)\cos\phi(t'',t')-2\frac{q''q'}{\sigma''\sigma'}\right.\right.\\&\quad+\frac{2q''}{\omega\sigma''}\int_{t'}^{t''}H(t)\sin\phi(t,t')\,\mathrm{d}t+\frac{2q'}{\omega\sigma'}\int_{t'}^{t''}H(t)\sin\phi(t'',t)\,\mathrm{d}t\\&\quad\left.\left.-\frac{2}{\omega^2}\int_{t'}^{t''}\mathrm{d}t\int_{t'}^{t}\mathrm{d}s\,H(t)H(s)\sin\phi(t'',t)\sin\phi(s,t')\right]\right\}.\end{aligned} \tag{3.24}$$

As expected this expression agrees with the one derived by Khandekar and Lawande (1979) by an explicit path integration of the time-dependent Lagrangian of the problem. Note that the present derivation does not require explicit path integration as the auxiliary time-independent problem involves only a free-particle propagator. Further, the general expression (3.24) contains the special case of the free and forced oscillator with a constant frequency or the free oscillator with a variable frequency.

As a second example we consider a time-dependent harmonic oscillator with a frequency $\Omega(t)$ acted on by an additional inverse quadratic potential g/q^2 where g is constant ($g>-\hbar^2/8$). Comparing

$$V(q,t)=\tfrac{1}{2}\Omega^2(t)q^2+g/q^2 \tag{3.25}$$

with the form (2.3) we have

$$\alpha = 0, \qquad F(q) = g/q^2 \tag{3.26}$$

while $\rho(t)$ is a solution of (3.2). According to (2.17) and (2.19*b*) the propagator $\bar{K}_0(\xi'', \tau''; \xi', \tau')$ takes the form

$$\bar{K}_0(\xi'', \tau''; \xi', \tau') = \int \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{\tau'}^{\tau''}\left[\frac{1}{2}\left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)^2 - \frac{g}{\xi^2}\right]\mathrm{d}\tau\right\}\mathscr{D}\xi(\tau), \tag{3.27}$$

where $\xi = q/\rho$. Writing

$$g = \tfrac{1}{2}\hbar^2(\gamma - \tfrac{1}{2})(\gamma + \tfrac{1}{2}), \qquad \gamma = \tfrac{1}{2}(1 + 8g/\hbar^2)^{1/2}, \tag{3.28}$$

we see that the propagator of (3.27) resembles the radial propagator of a three-dimensional free particle corresponding to an 'angular momentum' $(\gamma - \frac{1}{2})$. The latter has been derived by Peak and Inomata (1979) and has the form

$$\bar{K}_0(\xi'', \tau''; \xi', \tau') = (\xi'\xi'')^{1/2}\left(\frac{1}{\mathrm{i}\hbar(\tau'' - \tau')}\right)\exp\left(\frac{\mathrm{i}}{2\hbar}\frac{(\xi''^2 + \xi'^2)}{(\tau'' - \tau')}\right)I_\gamma\left(\frac{\xi''\xi'}{\mathrm{i}\hbar(\tau'' - \tau')}\right). \tag{3.29}$$

Inserting (3.29) in (2.19*a*) we obtain

$$K(q'', t''; q', t') = (c/\mathrm{i}\hbar)(q'q'')^{1/2}\exp[(\mathrm{i}/2\hbar)(aq''^2 + bq'^2)]I_\gamma(cq'q''/\mathrm{i}\hbar) \tag{3.30}$$

where the coefficients *a*, *b*, *c* are as defined in (3.6)–(3.8). When evaluated using the solution (3.12) for ρ, they are then given by (3.18)–(3.20). It therefore follows that the final expression for the propagator has the exact analytical form

$$\begin{aligned} K(q'', t''; q', t') &\\ = \left(\frac{\omega(q'q'')^{1/2}}{\mathrm{i}\hbar\sigma''\sigma'\sin\phi(t'', t')}\right)&\exp\left[\frac{\mathrm{i}}{2\hbar}\left(\frac{\dot{\sigma}''}{\sigma''}q''^2 - \frac{\dot{\sigma}'}{\sigma'}q'^2\right)\right]\\ &\times\exp\left\{\frac{\mathrm{i}\omega}{2\hbar}\left(\frac{q''^2}{\sigma''^2} + \frac{q'^2}{\sigma'^2}\right)\cot\phi(t'', t')\right\}\\ &\times I_\gamma\left(\frac{\omega q'' q'}{\mathrm{i}\hbar\sigma''\sigma'\sin\phi(t'', t')}\right) \qquad (0 < q', q'' < \infty) \end{aligned} \tag{3.31}$$

derived previously (Khandekar and Lawande 1975) by an explicit path integration technique.

We may remark that apart from the examples considered above, the evaluation of the associated propagator $\bar{K}_0$ is far from simple in the general case of arbitrary *F*. Nevertheless, removal of time dependence from the phase factor renders the problem time independent. The propagator $\bar{K}_0$ may then be computed by taking a recourse to a perturbative analysis (Feynman and Hibbs 1965). Alternatively, one may be able to solve the classical equation of motion corresponding to the reduced Lagrangian $\bar{L}_0$ to obtain the classical path. It is then possible to adopt a semi-classical expansion of $\bar{K}_0$ involving expansion of the action around the classical path (Dewitt-Morette 1976).

4. Conclusions

The major contribution of this paper is to show, without explicit path integration, that for a one-dimensional time-dependent system possessing an invariant quadratic in momentum the Feynman propagator is related to a propagator for an associated time-independent problem. If the latter propagator is known as in the case of the illustrative examples discussed, the former is completely evaluated. The role of the invariant and the equivalence of the present theory with that of Lewis and Reisenfeld is discussed. It will be interesting to generalise these results to systems possessing invariants more general than quadratic in p.

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